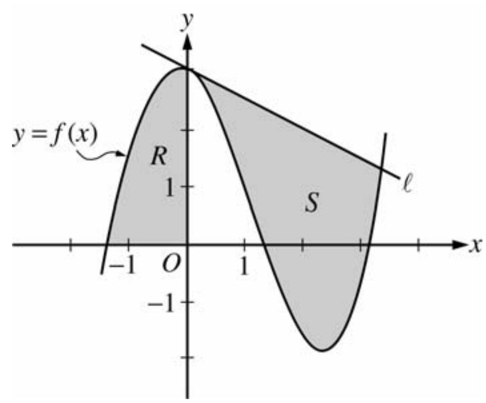
Question 1



Let f be the function given by $f(x) = \frac{x^3}{4} - \frac{x^2}{3} - \frac{x}{2} + 3\cos x$. Let R be the shaded region in the second quadrant bounded by the graph of f, and let S be the shaded region bounded by the graph of f and line ℓ, the line tangent to the graph of f at $x = 0$, as shown above.

(a) Find the area of R.

(b) Find the volume of the solid generated when R is rotated about the horizontal line $y = -2$.

(c) Write, but do not evaluate, an integral expression that can be used to find the area of S.

For $x < 0$, $f(x) = 0$ when $x = -1.37312$.
Let $P = -1.37312$.

(a) Area of $R = \int_P^0 f(x)\,dx = 2.903$

$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

(b) Volume $= \pi \int_P^0 \left((f(x) + 2)^2 - 4 \right) dx = 59.361$

$4 : \begin{cases} 1 : \text{limits and constant} \\ 2 : \text{integrand} \\ 1 : \text{answer} \end{cases}$

(c) The equation of the tangent line ℓ is $y = 3 - \frac{1}{2}x$.

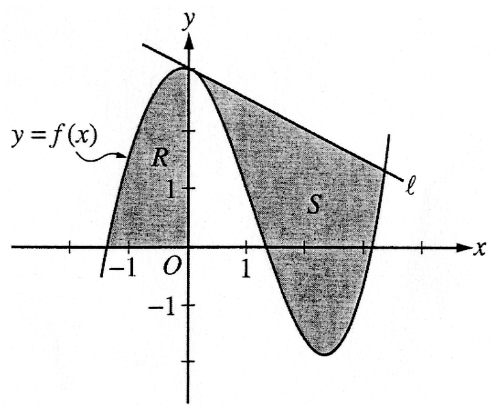
The graph of f and line ℓ intersect at $A = 3.38987$.

Area of $S = \int_0^A \left(\left(3 - \frac{1}{2}x \right) - f(x) \right) dx$

$3 : \begin{cases} 1 : \text{tangent line} \\ 1 : \text{integrand} \\ 1 : \text{limits} \end{cases}$

CALCULUS AB
SECTION II, Part A
Time—45 minutes
Number of problems—3

A graphing calculator is required for some problems or parts of problems.



Work for problem 1(a)

Area $R = \int_a^b f(x)\,dx$

$f(x) = 0 \Rightarrow \frac{x^3}{4} - \frac{x^2}{3} - \frac{x}{2} + 3\cos x = 0$

$x = a = -1.373$

$b = 0$

$R = \int_{-1.373}^{0} f(x)\,dx = 2.903$

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Continue problem 1 on page 5.

Work for problem 1(b)

$$V = \pi \int_{-1.373}^{0} \left[\left(f(x)+2\right)^2 - (2)^2 \right] dx = 59.361 \text{ unit}^3$$

Work for problem 1(c)

$$f'(x) = \frac{3x^2}{4} - \frac{2x}{3} - \frac{1}{2} - 3\sin x$$

$f'(0) =$ slope at $x = 0$

$$f'(0) = -\frac{1}{2}$$

$$f(0) = 3$$

$$y - 3 = -\frac{1}{2}(x - 0)$$

$$y = 3 - \frac{x}{2}$$

intersection point when $3 - \frac{x}{2} = f(x) \Rightarrow x = 3.390$

$$\text{Area } S = \int_{0}^{3.390} \left[3 - \frac{x}{2} - \left(\frac{x^3}{4} - \frac{x^2}{3} - \frac{x}{2} + 3\cos x \right) \right] dx$$

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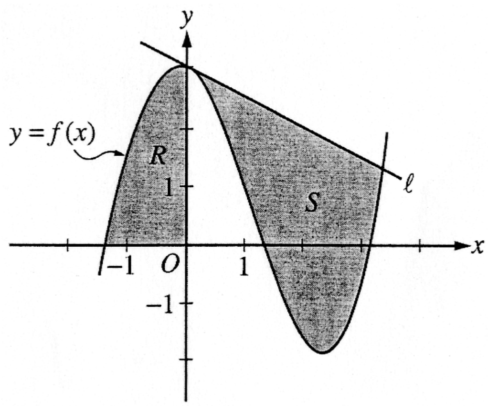
1B

CALCULUS AB
SECTION II, Part A
Time—45 minutes
Number of problems—3

A graphing calculator is required for some problems or parts of problems.



Work for problem 1(a)

$\frac{x^3}{4} - \frac{x^2}{3} - \frac{x}{2} + 3\cos x = 0$

$x = -1.37312$

$\int_{-1.37312}^{0} \frac{x^3}{4} - \frac{x^2}{3} - \frac{x}{2} + 3\cos x \, dx$

$= 2.90309.$

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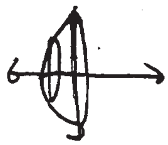
Continue problem 1 on page 5.

Work for problem 1(b)



$r = \frac{x^3}{4} - \frac{x^2}{3} - \frac{x}{2} + 3\cos x$

$$\int_{-1.37312}^{0} \pi r^2 \, dx = \int_{-1.37312}^{0} \pi \left(\frac{x^3}{4} - \frac{x^2}{3} - \frac{x}{2} + 3\cos x\right)^2 dx = 22.880$$

Work for problem 1(c)

$f'(0) = -\frac{1}{2}$ $\quad y = -\frac{1}{2}x + 6$

$y = -\frac{1}{2}x + 3$ is ℓ.

Intesect $-\frac{1}{2}x + 3 = \frac{x^3}{4} - \frac{x^2}{3} - \frac{x}{2} + 3\cos$

$x = 3.38987 \quad y = 1.30507$

$$\int_{0}^{3.38987} \left| \left(-\frac{1}{2}x + 3\right) - \left(\frac{x^3}{4} - \frac{x^2}{3} - \frac{x}{2} + 3\cos x\right) \right| dx$$

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1C

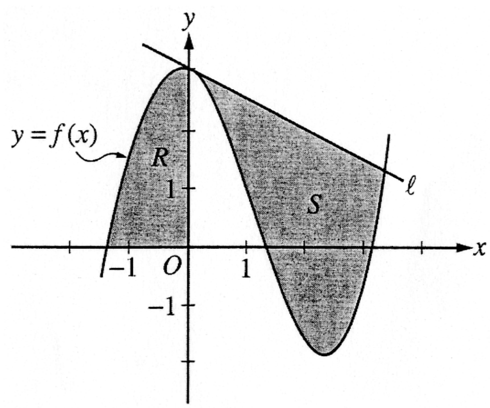
CALCULUS BC

SECTION II, Part A

Time—45 minutes

Number of problems—3

A graphing calculator is required for some problems or parts of problems.



Work for problem 1(a)

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bounds extend from

-1.37312 to 0

$$\therefore A_R = \int_{-1.37312}^{0} \frac{x^3}{4} - \frac{x^2}{3} - \frac{x}{2} + 3\cos x \, dx$$

$$\boxed{= 2.903}$$

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Continue problem 1 on page 5.

1C

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Work for problem 1(b)

$$\pi \int_{-1.373/2}^{0} \left(\frac{x^3}{4} - \frac{x^2}{3} - \frac{x}{2} + 3\cos x\right)^2 - 2^2$$

$$\boxed{= 1.79\pi}$$

Work for problem 1(c)

$$A_S = \int_0^{3.1} 1 - f(x)\, dx$$

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Question 1

Overview

This problem presented students with two regions. Region R was bounded in the second quadrant by a graph and the two axes. Region S was bounded by the graph and the line tangent to the graph at one point. Students needed to use integration to find two areas and a volume. In order to answer parts (a) and (b), students also had to find a zero of the function to obtain bounding values for region R. Part (a) asked students to find the area of R. Part (b) asked students to find the volume of the solid generated by rotating R about a horizontal line. In part (c) students had to find the equation of the tangent line and the x-coordinate of a point of intersection of the line and the graph in order to write an integral expression for the area of S.

Sample: 1A
Score: 9

The student earned all 9 points.

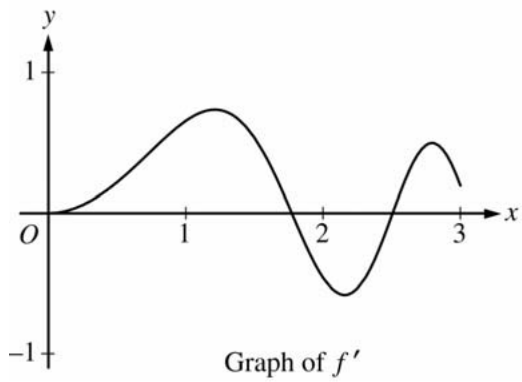
Sample: 1B
Score: 6

The student earned 6 points: 2 points in part (a), 1 point in part (b), and 3 points in part (c). The work in part (a) is correct. In part (b) the student earned the limits and constant point. The student writes an integral for rotation about the x-axis and does not consider the horizontal line $y = -2$. Because of this error in the integrand, the student was not eligible for the answer point. The work in part (c) earned all 3 points.

Sample: 1C
Score: 4

The student earned 4 points: 2 points in part (a) and 2 points in part (b). The work in part (a) is correct. In part (b) the student earned the limits and constant point. The student earned 1 of the 2 integrand points. The first term of the integrand is incorrect since 2 was not added to $f(x)$. Because of this error in the integrand, the student was not eligible for the answer point. In part (c) the equation of the tangent line is not found, so the tangent line point was not earned. Since an equation for a tangent line was not found, the student could not earn the integrand point. In addition, the student did not earn the limits point for estimating the intersection point from the given graph.

Question 2



Let f be the function defined for $x \ge 0$ with $f(0) = 5$ and f', the first derivative of f, given by $f'(x) = e^{(-x/4)} \sin\left(x^2\right)$. The graph of $y = f'(x)$ is shown above.

(a) Use the graph of f' to determine whether the graph of f is concave up, concave down, or neither on the interval $1.7 < x < 1.9$. Explain your reasoning.

(b) On the interval $0 \le x \le 3$, find the value of x at which f has an absolute maximum. Justify your answer.

(c) Write an equation for the line tangent to the graph of f at $x = 2$.

(a) On the interval $1.7 < x < 1.9$, f' is decreasing and thus f is concave down on this interval.

$2 : \begin{cases} 1 : \text{answer} \\ 1 : \text{reason} \end{cases}$

(b) $f'(x) = 0$ when $x = 0, \sqrt{\pi}, \sqrt{2\pi}, \sqrt{3\pi}, \ldots$
On $[0, 3]$ f' changes from positive to negative only at $\sqrt{\pi}$. The absolute maximum must occur at $x = \sqrt{\pi}$ or at an endpoint.

$f(0) = 5$

$f\left(\sqrt{\pi}\right) = f(0) + \int_0^{\sqrt{\pi}} f'(x)\,dx = 5.67911$

$f(3) = f(0) + \int_0^3 f'(x)\,dx = 5.57893$

This shows that f has an absolute maximum at $x = \sqrt{\pi}$.

$3 : \begin{cases} 1 : \text{identifies } \sqrt{\pi} \text{ and 3 as candidates} \\ \qquad\quad \text{- or -} \\ \quad \text{indicates that the graph of } f \\ \quad \text{increases, decreases, then increases} \\ 1 : \text{justifies } f\left(\sqrt{\pi}\right) > f(3) \\ 1 : \text{answer} \end{cases}$

(c) $f(2) = f(0) + \int_0^2 f'(x)\,dx = 5.62342$

$f'(2) = e^{-0.5} \sin(4) = -0.45902$

$y - 5.623 = (-0.459)(x - 2)$

$4 : \begin{cases} 2 : f(2) \text{ expression} \\ \quad 1 : \text{integral} \\ \quad 1 : \text{including } f(0) \text{ term} \\ 1 : f'(2) \\ 1 : \text{equation} \end{cases}$

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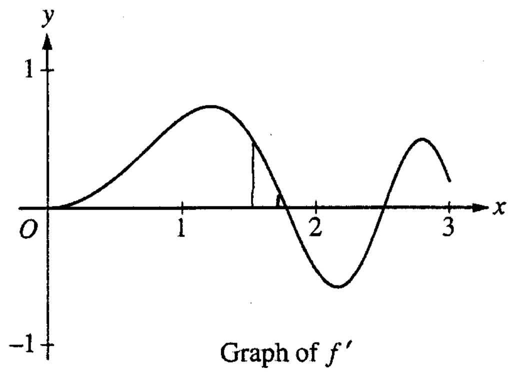


Graph of f'

Work for problem 2(a)

if $f''(x) < 0$ on some interval, then the graph of f is concave down on this interval

if $f''(x) > 0$ on some interval, then the graph is concave ~~down~~ up on this interval

the interval where $f''(x) < 0$ determines also where $f'(x)$ is decreasing, and where $f''(x) > 0$, $f'(x)$ is increasing.

From the graph of $f'(x)$, it can be seen that on the interval $(1.7; 1.9)$, $f'(x)$ is decreasing, therefore, the graph of $f(x)$ will be concave down.

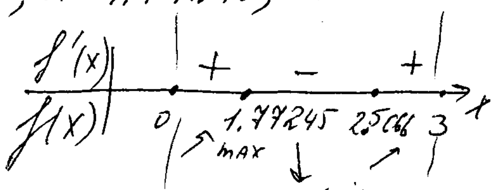
Continue problem 2 on page 7.

Work for problem 2(b)

First, find the critical points. Critical points are where $f'(x)=0$ or $f'(x)\not\exists$

~~From the graph, it can be seen that $f'(x)=0$ at $x=0$, $x=$~~

$f'(x)=0$: $e^{(-x/4)}\sin(x^2)=0$: $x=0$, $x=1.77245$, $x=$



Since $f'(x)$ changes from positive to negative at $x=1.77245$, $f(1.77245)$ is a relative maximum. As $f'(x)$ changes from negative to positive at $x=2.5066$, $f(2.5066)$ is a relative minimum. Since $f'(x)$ is positive on $(0; 1.77245)$, $f(x)$ increases on this interval and $f(1.77245)>f(0)$

So, the absolute maximum is either at $x=1.77245$ or at the upper endpoint $x=3$

$f(1.77245)=f(0)+\int_0^{1.77245} e^{(-x/4)}\sin(x^2)dx=5+0.6791141=5.6791141$

$f(3)=f(0)+\int_0^3 e^{(-x/4)}\cdot\sin(x^2)dx=5+0.5789285=5.5789285$

$f(1.77245)>f(3)$, Thus, $x=1.77245$ is the value of x where f has an absolute maximum

Work for problem 2(c)

$x_0=2$

$y_t=f(x_0)+f'(x_0)(x-x_0)$

$f(2)=f(0)+\int_0^2 e^{(-x/4)}\cdot\sin(x^2)dx=5+0.6234267=5.6234267$

$f'(2)=-0.4590239$

$y_t=5.6234267-0.4590239(x-2)$

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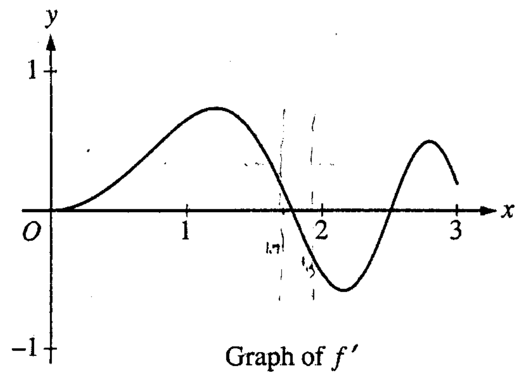

Graph of f'

Work for problem 2(a)

Step 1: Find the inflection point between 1.5 and 2 approx

$$f'(x) = e^{(-x/4)} \sin(x^2)$$

$$f''(x) = -\frac{e^{(-x/4)}}{4} \sin(x^2) + e^{(-x/4)} (2x)\cos(x^2)$$

inflection $\Rightarrow$ $f''(x) = 0$

No inflection pts found between 1.7 & 1.9

$\Rightarrow$ either concave up or down

$\Rightarrow$ $f''(1.7) = -2.1935 < 0$ concave down

to check

$f''(1.9) = -2.0384 < 0$ concave down

Over the interval $1.7 < x < 1.9$ the graph of f is CONCAVE DOWN

Continue problem 2 on page 7.

Work for problem 2(b)

For f to have an absolute maximum in the interval $0 \le x \le 3$ there has to exist a point at which f' is ZERO & f'' is < 0 such that it is concave down

From the graph $\Rightarrow$ f' is zero at 3 pts $x = 0$, $x = 1.7724538509$, $x = 2.5066282746$

$\Rightarrow f''(0) = 0$ inflection

$f''(1.7724538509) = -2.276 < 0 \Leftarrow$ Max

$f''(2.5066282746) = 2.679 > 0 \Rightarrow$ Min

x at absolute max. is $x = 1.7724538509$

Work for problem 2(c)

$f'(2) = -0.459$

$x = 2 \quad y = \int_0^2 e^{(-x/4)} \sin(x^2)\, dx$

$y|_{x=2} = -0.6234 + 5 = 4.377$

$y - 4.377 = -0.459(x - 2)$

$y = -0.459x + 5.295$

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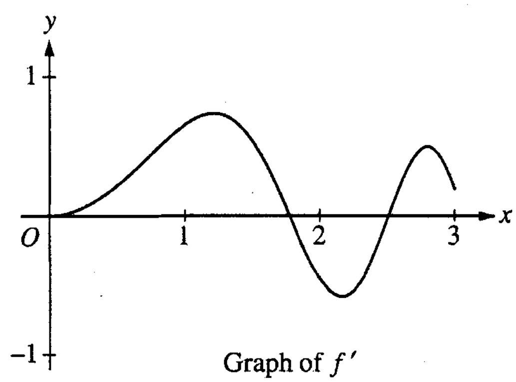
2C



Graph of f'

Work for problem 2(a)

f' is decreasing, therefore $f''(x) < 0$, the function $f(x)$ is concave down

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Continue problem 2 on page 7.

Work for problem 2(b)

f has a relative max when f' changes its sing from + to -

$X = 1,772$

Then we compare the values of $f(x)$ at $x = 1.772$, $X=0$, $x=3$

~~[illegible]~~ but as the value of $f'(x)$ on $(0; 1,772) > 0$, that means that $f(1.772) > f(0)$

Work for problem 2(c)

$y = f'(x_0)(x - x_0) + f(x_0)$

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Question 2

Overview

This problem presented students with the graph and the symbolic formula of the derivative of a function f. It explored their understanding of the relationship between the behavior of the derivative and the function. In part (a) students had to know how to determine the concavity of the graph of f by observing where the graph of f' was increasing or decreasing. Part (b) asked students to find and justify the location of the absolute maximum of f over the given closed interval. It was expected that students would use the graphing calculator to find the appropriate critical point for a local maximum and to evaluate f at that critical point and the endpoints using definite integrals. However, students could also find the relevant critical point by hand and compare the values of f at that critical point and at the endpoints using signed areas. Part (c) required knowing the relationship between the values of f' and the slope of the line tangent to the graph of f. It also required the use of a definite integral to compute $f(2)$.

Sample: 2A
Score: 9

The student earned all 9 points. The discussion and conclusion in part (a) is correct. In part (b) the student identifies the appropriate candidates, uses a sign chart to summarize the behavior of the function and the derivative on the interval $[0, 3]$, and then writes a correct explanation for the location of the absolute maximum. The work in part (c) is correct.

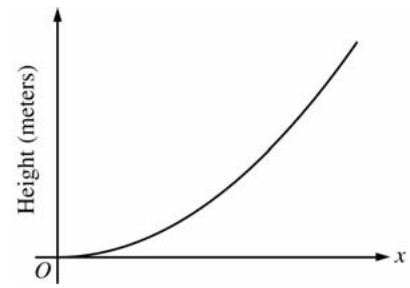
Sample: 2B
Score: 6

The student earned 6 points: 2 points in part (a), 1 point in part (b), and 3 points in part (c). In part (a) the student computes the correct second derivative and determines that there are no inflection points between $x = 1.7$ and $x = 1.9$. The student then checks the sign of the second derivative at the two endpoints of the given interval. This is sufficient to make the correct conclusion since there is no inflection point in the interval. In part (b) the student only justifies a local maximum at $x = 1.772$ and thus only earned the third point. In part (c) the student sets up a definite integral for the change in f over the interval from $x = 0$ to $x = 2$ and then adds $f(0)$ in an attempt to compute $f(2)$. This earned the first 2 points. The student also computes the correct derivative value, which earned the third point. The student did not earn the last point because of the error in the evaluation of the integral.

Sample: 2C
Score: 4

The student earned 4 points: 2 points in part (a) and 2 points in part (b). The work in part (a) earned both points. In part (b) the student justifies $x = 1.772$ as the location of a relative maximum. The student indicates the need to compare the value of $f(x)$ at $x = 1.772$ with the values at both endpoints but then only deals with the endpoint at $x = 0$ and not the endpoint at $x = 3$. The student thus earned only the first and third points in part (b). The expression for the tangent line in part (c) is considered a formula and earned no points.

Question 3



The figure above is the graph of a function of x, which models the height of a skateboard ramp. The function meets the following requirements.

(i) At $x = 0$, the value of the function is 0, and the slope of the graph of the function is 0.
(ii) At $x = 4$, the value of the function is 1, and the slope of the graph of the function is 1.
(iii) Between $x = 0$ and $x = 4$, the function is increasing.

(a) Let $f(x) = ax^2$, where a is a nonzero constant. Show that it is not possible to find a value for a so that f meets requirement (ii) above.

(b) Let $g(x) = cx^3 - \frac{x^2}{16}$, where c is a nonzero constant. Find the value of c so that g meets requirement (ii) above. Show the work that leads to your answer.

(c) Using the function g and your value of c from part (b), show that g does not meet requirement (iii) above.

(d) Let $h(x) = \frac{x^n}{k}$, where k is a nonzero constant and n is a positive integer. Find the values of k and n so that h meets requirement (ii) above. Show that h also meets requirements (i) and (iii) above.

(a) $f(4) = 1$ implies that $a = \frac{1}{16}$ and $f'(4) = 2a(4) = 1$ implies that $a = \frac{1}{8}$. Thus, f cannot satisfy (ii).	$2 : \begin{cases} 1 : a = \frac{1}{16} \text{ or } a = \frac{1}{8} \\ 1 : \text{shows } a \text{ does not work} \end{cases}$
(b) $g(4) = 64c - 1 = 1$ implies that $c = \frac{1}{32}$. When $c = \frac{1}{32}$, $g'(4) = 3c(4)^2 - \frac{2(4)}{16} = 3\left(\frac{1}{32}\right)(16) - \frac{1}{2} = 1$	1 : value of c
(c) $g'(x) = \frac{3}{32}x^2 - \frac{x}{8} = \frac{1}{32}x(3x-4)$ $g'(x) < 0$ for $0 < x < \frac{4}{3}$, so g does not satisfy (iii).	$2 : \begin{cases} 1 : g'(x) \\ 1 : \text{explanation} \end{cases}$
(d) $h(4) = \frac{4^n}{k} = 1$ implies that $4^n = k$. $h'(4) = \frac{n4^{n-1}}{k} = \frac{n4^{n-1}}{4^n} = \frac{n}{4} = 1$ gives $n = 4$ and $k = 4^4 = 256$. $h(x) = \frac{x^4}{256} \Rightarrow h(0) = 0.$ $h'(x) = \frac{4x^3}{256} \Rightarrow h'(0) = 0$ and $h'(x) > 0$ for $0 < x < 4$.	$4 : \begin{cases} 1 : \frac{4^n}{k} = 1 \\ 1 : \frac{n4^{n-1}}{k} = 1 \\ 1 : \text{values for } k \text{ and } n \\ 1 : \text{verifications} \end{cases}$

Work for problem 3(a)

according to (ii), $f(4)=1$, $f'(4)=1$

$f(x)=ax^2 \rightarrow 16a=1 \quad a=\frac{1}{16}$

$f'(x)=2ax \rightarrow 8x=1 \quad a=\frac{1}{8}$

$\frac{1}{16} \neq \frac{1}{8}$

$\therefore$ it's impossible to find a value for a so that f meets requirement (ii).

Work for problem 3(b)

according to (ii), $g(4)=1$, $g'(4)=1$

$g(x)=cx^3-\frac{x^2}{16} \rightarrow 64c-\frac{16}{16}=64c-1=1 \qquad c=\frac{1}{32}$

$g'(x)=3cx^2-\frac{1}{8}x \rightarrow 3\cdot 16\cdot c-\frac{1}{2}=48c-\frac{1}{2}=1 \qquad c=\frac{1}{32}$

$\therefore c=\frac{1}{32}$

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Continue problem 3 on page 9

Work for problem 3(c)

$g'(x) = \frac{3}{32}x^2 - \frac{1}{8}x = \frac{3}{32}x(x - \frac{4}{3})$

$\therefore$ $x < 0$: $g'(x) > 0$, $g(x)$ increasing

$0 < x < \frac{4}{3}$: $g'(x) < 0$, $g(x)$ decreasing

$\frac{4}{3} < x$: $g'(x) > 0$, $g(x)$ increasing

$g(x)$ do not increase when $0 < x < \frac{4}{3}$. So it does not meet requirement (iii)

Work for problem 3(d)

according to (ii), $h(4) = 1$, $h'(4) = 1$

$h(x) = \frac{x^n}{k} \rightarrow \frac{4^n}{k} = 1$

$h'(x) = \frac{n}{k} \cdot x^{n-1} \rightarrow \frac{n}{k} \cdot 4^{n-1} = 1$

$4^n = k$, $4^{n-1} \cdot n = k$

$\therefore n = 4$ $k = 256$

$\therefore h(x) = \frac{x^4}{256}$

$h(0) = 0$, $h'(0) = 0$ $\rightarrow$ meet requirement (i)

$h'(x) = \frac{4}{256} \cdot x^3 = \frac{1}{64}x^3$ $x > 0$, $h'(x) > 0$ $\therefore h(x)$ increasing $\rightarrow$ meet requirement (iii).

END OF PART A OF SECTION II
IF YOU FINISH BEFORE TIME IS CALLED, YOU MAY CHECK YOUR WORK ON PART A ONLY. DO NOT GO ON TO PART B UNTIL YOU ARE TOLD TO DO SO.

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Work for problem 3(a)

$f(x)=ax^2$

$f'(x)=2ax$

to satisfy (ii), $x=4$

$f(4)=1$

$f'(4)=1$

$f(x)=16a=1$

$a=\frac{1}{16}$

$f'(x)=2\cdot 4a=1$

$a=\frac{1}{8}$

- There is no value a that satisfies requirement (ii)

Work for problem 3(b)

$g(x)=cx^3-\frac{x^2}{16}$

$g'(x)=3cx^2-\frac{x}{8}$

$g(4)=64c-1=1$ $c=\frac{1}{32}$

$g'(4)=48c-.5=1$ $c=\frac{1}{32}$

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Continue problem 3 on page 9.

Work for problem 3(c)

$g(x) = \frac{1}{32}x^3 - \frac{x^2}{16}$ $\quad g'(0) = 0$

$g'(x) = \frac{3}{32}x^2 - \frac{x}{8} = 0$ $\quad g'(4) = 1$

$x\left(\frac{3}{32}x - \frac{1}{8}\right) = 0$

$x = 0 \quad x = 1.333$

Because $g'(0) = 0$, $g(x)$ is not increasing at $x=0$, thus it does not satisfy requirement (iii)

Work for problem 3(d)

$h(x) = \frac{x^n}{k}$ $\quad \frac{4^n}{k} = 1 \quad 4^n = k$

$h'(x) = \frac{nx^{n-1}}{k}$ $\quad \frac{n4^{n-1}}{k} = 1 \quad n4^{n-1} = k$

$4^n = n4^{n-1}$

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END OF PART A OF SECTION II

IF YOU FINISH BEFORE TIME IS CALLED, YOU MAY CHECK YOUR WORK ON PART A ONLY. DO NOT GO ON TO PART B UNTIL YOU ARE TOLD TO DO SO.

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Work for problem 3(a)

$f(x) = ax^2$

$y = ax^2$

$1 = 16a$

$a = \frac{1}{16}$

$y = \frac{1}{16}x^2$

$a = \frac{1}{16}$

Work for problem 3(b)

$x = 4$
$y = 1$

$1 = 64c - 1$

$2 = 64c$

$c = \frac{1}{32}$

Continue problem 3 on page 9

3C

Work for problem 3(c)

$g(x) = \frac{x^3}{32} - \frac{x^2}{16}$

$= \frac{x^3 - 2x^2}{32}$

$x = 0$
$y = 0$

$x = 1$
$y = -\frac{1}{32}$

$x = 2$
$y = 0$

$x = 3$
$y = 0$

$x = 4$
$y = 1$

Work for problem 3(d)

$h(x) = \frac{x^n}{k}$

$1 = \frac{4^n}{k}$

$k = 4^n.$

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END OF PART A OF SECTION II

IF YOU FINISH BEFORE TIME IS CALLED, YOU MAY CHECK YOUR WORK ON PART A ONLY. DO NOT GO ON TO PART B UNTIL YOU ARE TOLD TO DO SO.

Question 3

Overview

This problem presented three requirements that had to be satisfied by the graph of a function modeling the height of a skateboard ramp. Students were asked to investigate three families of functions that might be used for such a model. In part (a) they were asked to show that no quadratic of the form ax^2 would satisfy the second requirement. In part (b) they were asked to find the coefficient c for which the cubic $cx^3 - \frac{x^2}{16}$ would meet the second requirement, but then show in part (c) that the cubic with this value of c does not meet the third requirement. Finally, in part (d) students were asked to find the values of n and k for which the power function $\frac{x^n}{k}$ would meet all three requirements.

Sample: 3A
Score: 9

The student earned all 9 points.

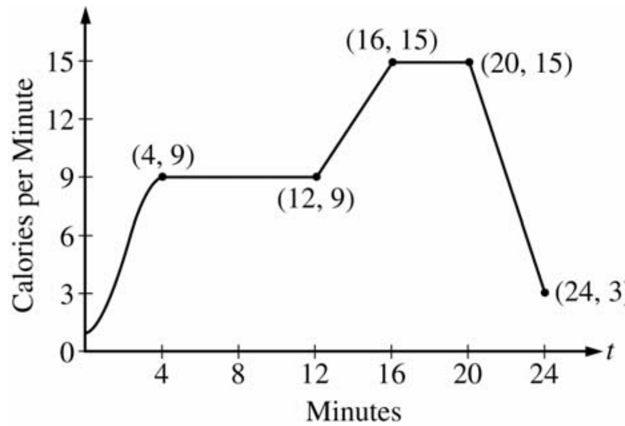
Sample: 3B
Score: 6

The student earned 6 points: 2 points in part (a), 1 point in part (b), 1 point in part (c), and 2 points in part (d). The student's work is correct in parts (a) and (b). In part (c) the student earned 1 point for finding the derivative of g. The student does not explain why g is not increasing between $x = 0$ and $x = 4$ and so did not earn the second point in this part. In part (d) the student sets up correct equations to find n and k, earning 1 point for each equation, but does not find n or k and thus cannot show that the function h meets requirements (i) and (iii).

Sample: 3C
Score: 3

The student earned 3 points: 1 point in part (a), 1 point in part (b), and 1 point in part (d). In part (a) the student finds the value of a for which $f(4) = 1$, which earned the first point, but fails to show that this value of a does not work to meet requirement (ii). In part (b) the student uses the information about g to find the desired value of c. In part (c) the student's calculations of the values of the function g at integer values of x earned no points (and the value at $x = 3$ is incorrect). However, both points could have been earned in part (c) with those calculations if the student had gone on to observe that the value of y at $x = 1$ is less than the value of y at $x = 0$, and hence the function g is not increasing on the interval $0 \le x \le 4$. In part (d) the student earned 1 point for using the information about $h(4)$ to write an equation for n and k but has no other work.

Question 4



The rate, in calories per minute, at which a person using an exercise machine burns calories is modeled by the function f. In the figure above, $f(t) = -\frac{1}{4}t^3 + \frac{3}{2}t^2 + 1$ for $0 \le t \le 4$ and f is piecewise linear for $4 \le t \le 24$.

(a) Find $f'(22)$. Indicate units of measure.

(b) For the time interval $0 \le t \le 24$, at what time t is f increasing at its greatest rate? Show the reasoning that supports your answer.

(c) Find the total number of calories burned over the time interval $6 \le t \le 18$ minutes.

(d) The setting on the machine is now changed so that the person burns $f(t) + c$ calories per minute. For this setting, find c so that an average of 15 calories per minute is burned during the time interval $6 \le t \le 18$.

(a) $f'(22) = \dfrac{15-3}{20-24} = -3$ calories/min/min

1 : $f'(22)$ and units

(b) f is increasing on $[0, 4]$ and on $[12, 16]$.

On $(12, 16)$, $f'(t) = \dfrac{15-9}{16-12} = \dfrac{3}{2}$ since f has constant slope on this interval.

On $(0, 4)$, $f'(t) = -\frac{3}{4}t^2 + 3t$ and

$f''(t) = -\frac{3}{2}t + 3 = 0$ when $t = 2$. This is where f' has a maximum on $[0, 4]$ since $f'' > 0$ on $(0, 2)$ and $f'' < 0$ on $(2, 4)$.

On $[0, 24]$, f is increasing at its greatest rate when $t = 2$ because $f'(2) = 3 > \frac{3}{2}$.

$$4 : \begin{cases} 1 : f' \text{ on } (0, 4) \\ 1 : \text{shows } f' \text{ has a max at } t = 2 \text{ on } (0, 4) \\ 1 : \text{shows for } 12 < t < 16,\ f'(t) < f'(2) \\ 1 : \text{answer} \end{cases}$$

(c) $\int_6^{18} f(t)\,dt = 6(9) + \frac{1}{2}(4)(9+15) + 2(15)$
$= 132$ calories

$$2 : \begin{cases} 1 : \text{method} \\ 1 : \text{answer} \end{cases}$$

(d) We want $\frac{1}{12}\int_6^{18} (f(t) + c)\,dt = 15$.

This means $132 + 12c = 15(12)$. So, $c = 4$.

OR

Currently, the average is $\dfrac{132}{12} = 11$ calories/min.

Adding c to $f(t)$ will shift the average by c.
So $c = 4$ to get an average of 15 calories/min.

$$2 : \begin{cases} 1 : \text{setup} \\ 1 : \text{value of } c \end{cases}$$

NO CALCULATOR ALLOWED

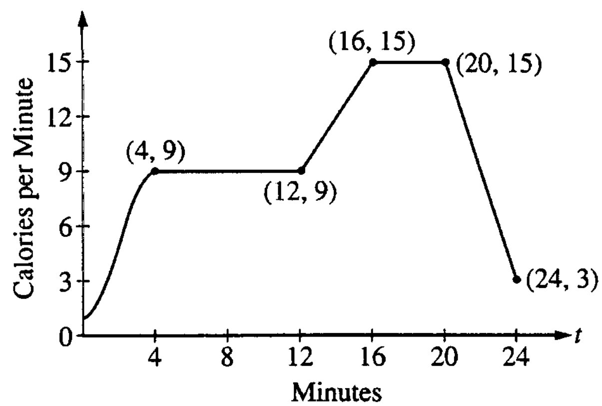
CALCULUS AB

SECTION II, Part B

Time—45 minutes

Number of problems—3

No calculator is allowed for these problems.



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Work for problem 4(a)

$f'(22)$ = Gradient of straight line from $t=20$ to $t=24$

$= \frac{3-15}{24-20}$

$= \frac{-12}{4}$

$= -3$ calories/minute^2

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Continue problem 4 on page 11.

NO CALCULATOR ALLOWED

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Work for problem 4(b)

From graph, we see that f is only increasing in the intervals $0 \leq t \leq 4$ and $12 \leq t \leq 16$

Rate of increment for $12 \leq t \leq 16$

$= \frac{15-9}{16-12}$

$= \frac{6}{4} = 1.5$

For $0 \leq t \leq 4$,

$f'(t) = -\frac{3}{4}t^2 + 3t$

$f''(t) = -\frac{3}{2}t + 3$

At greatest rate of increment, $f''(t) = 0 \Rightarrow t = 2$

$f'''(t) = -\frac{3}{2} < 0 \Rightarrow$ At $t = 2$, rate of increment is greatest and not smallest

$f'(2) = -\frac{3}{4}(2)^2 + 3(2) = 3 > 1.5$

$\therefore f$ is increasing at its greatest rate ~~than~~ at $t = 2$

Work for problem 4(c)

~~For 6≤t≤12~~ ~~For 6≤t≤12~~,

Total number of calories burned $= \int_6^{18} f(t)\,dt$

~~$= \int_6^{12} 9\,dt + \int_{12}^{16}$~~

$= 9(12-6) + \frac{1}{2}(9+15)(16-12) + 15(18-16)$

$= 54 + 48 + 30 = 132$ calories

Work for problem 4(d)

Before setting is changed, average calories in $6 \leq t \leq 18$

$= \frac{1}{18-6}\int_6^{18} f(t)\,dt = \frac{132}{12} = 11$ calories

Now, $\frac{1}{18-6}\int_6^{18}[f(t)+c]\,dt = 15$

$\Rightarrow \frac{1}{12}\int_6^{18} f(t)\,dt + \frac{1}{12}\int_6^{18} c\,dt = 15$

$\Rightarrow \frac{1}{12}[cx]_6^{18} = 15 - 11 = 4$

$\frac{1}{12}(18-6)c = 4$

$\Rightarrow c = 4$

GO ON TO THE NEXT PAGE.

 4B

NO CALCULATOR ALLOWED

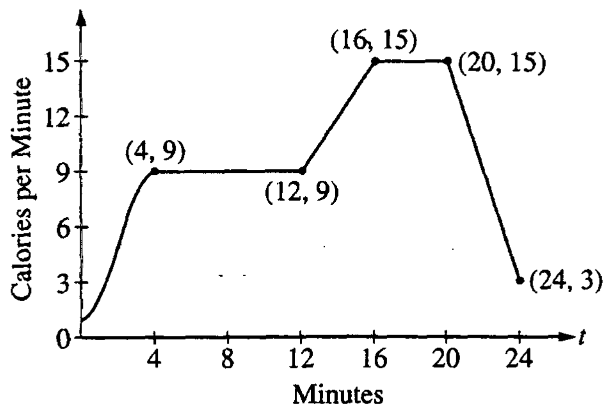
CALCULUS BC

SECTION II, Part B

Time—45 minutes

Number of problems—3

No calculator is allowed for these problems.



Work for problem 4(a)

$$f'(22) = \frac{f(24)-f(20)}{24-20} = \frac{3-15}{4} = -\frac{12}{4} = -3$$

$\therefore$ $-3 \text{ cal}/\text{min}^2$.

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Continue problem 4 on page 11

NO CALCULATOR ALLOWED

Work for problem 4(b)

i) $0 \le t \le 4$, $f'(t) = -\frac{3}{4}t^2 + 3t$

$= -\frac{3}{4}(t^2 - 4t + 4) + 3$

$= -\frac{3}{4}(t-2)^2 + 3$

ii) $f'(t) = 0$ for $4 \le t < 12$

iii) $f'(t) = \frac{15-9}{16-12} = \frac{6}{4} = \frac{3}{2}$ for $12 \le t < 16$

iv) $f'(t) = 0$ for $16 \le t < 20$

v) $f'(t) < 0$ for $20 \le t < 24$

∴ $f'(t)$ is the greatest when $t=2$, which also means that $f(t)$ is increasing at its greatest rate

∴ $t = 2$.

Work for problem 4(c)

i) $6 \le t < 12$ $f(t) = 9$: $6 \times 9 = 54$

ii) $12 \le t < 16$ $\frac{1}{2} \times (9+15) \times 4 = 48$

iii) $16 \le t \le 18$ $f(t) = 15$ $2 \times 15 = 30$

∴ $54 + 48 + 30 = 132$ ∴ 132 calories

Work for problem 4(d)

$132 + c \times (18-6) = 132 + 12c$

Since $\frac{132 + 12c}{12} = 15$, $c = \frac{180 - 132}{12} = 4$

∴ $c = 4$

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NO CALCULATOR ALLOWED

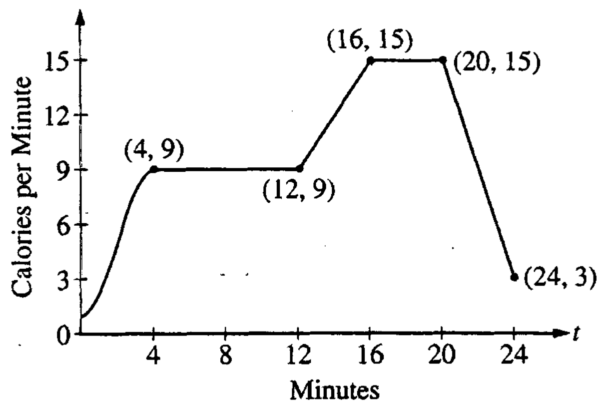
CALCULUS AB

SECTION II, Part B

Time—45 minutes

Number of problems—3

No calculator is allowed for these problems.



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Work for problem 4(a)

function for $20 \le x \le 24$

$m = \frac{3-15}{24-20}$ $y - 3 = -3(x - 24)$

$\frac{-12}{4}$ $y = -3x + 75$

$m = -3$

$f(x) = -3x + 75$

$f'(22) = -3$ calories/minutes^2

Continue problem 4 on page 11.

Work for problem 4(b)

function increases : at $0 \le x < 4$, $12 \le x \le 16$

$f(t) = -\frac{1}{4}t^3 + \frac{3}{2}t^2 + 1$

$f'(t) = -\frac{3}{4}t^2 + 3t$

$f''(t) = -\frac{3}{2}t + 3 = 0$

$t = 2$

maximum for $f'(t)$ at $t = 2$

$f'(2) = -\frac{3}{4}(4) + 6$

$= 3$

$m = \frac{15-9}{16-12} = \frac{3}{2}$

$f(x) = y - 15 = \frac{3}{2}(x-16)$

$f(x) = \frac{3}{2}x - 9$

$f'(x) = \frac{3}{2}$

Work for problem 4(c)

Calories total $= \int_6^{12} 9\,dx + \int_{12}^{16} (\frac{3}{2}x - 9)\,dx + \int_{16}^{18} 15\,dx$

$9x\Big]_6^{12} + \frac{3}{4}x^2 - 9\Big]_{12}^{16} + 15x\Big]_{16}^{18}$

$105 - 54 + 183 - 99 + 270 - 240$

$54 + 30 + 84$

164 calories burned

Work for problem 4(d)

$15 = \frac{\int_6^{18} (f(x) + c)\,dx}{12}$

$180 = \int_6^{18} f(x)\,dx + \int_6^{18} c\,dx$

Question 4

Overview

This problem presented students with a piecewise-defined function f that modeled the rate at which a person using an exercise machine burns calories. The graph of f consisted of a cubic part and a part that was piecewise linear. In part (a) students were asked to find $f'(22)$, which required them to recognize the relationship between this value and the slope of one of the line segments in the graph of f. It was also important to use correct units. For part (b) students had to consider the two parts of the graph where f was increasing and determine the time when f was increasing at its greatest rate. In part (c) students had to use a definite integral to find the total number of calories burned over a given time interval. The evaluation of the definite integral could be done using geometry since the graph over the given time interval consisted of two line segments, one of which was horizontal. For part (d) students were expected to use the value of their integral from part (c) to work with the average value of the function f shifted up by c calories per minute.

Sample: 4A
Score: 9

The student earned all 9 points.

Sample: 4B
Score: 6

The student earned 6 points: 1 point in part (a), 1 point in part (b), 2 points in part (c), and 2 points in part (d). The student shows correct work for parts (a), (c), and (d). In part (b) the first derivative is correct and the student earned the first point. The student neither explains why $f'(t)$ has a maximum at 2 nor states the value of $f'(2)$. A proper reasoning for the final answer is not given, and the student did not earn any more points.

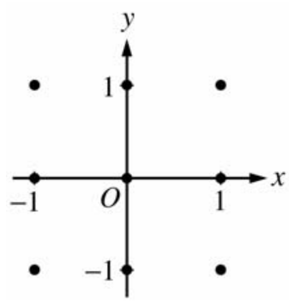
Sample: 4C
Score: 4

The student earned 4 points: 1 point in part (a), 1 point in part (b), 1 point in part (c), and 1 point in part (d). The work in part (a) is correct. In part (b) the first derivative is correct, so the student earned the first point. The student neither explains why $f'(t)$ has a maximum at 2 in the interval $0 \le t \le 4$ nor shows a comparison among the values of $f'(t)$. The student does not provide a final answer and did not earn any more points. In part (c) the setup is correct. The antiderivative of the second integral is incorrect, so the student did not earn the answer point. In part (d) the setup is correct, but the student does not finish the problem and could not earn the second point.

Question 5

Consider the differential equation $\frac{dy}{dx} = (y-1)^2 \cos(\pi x)$.

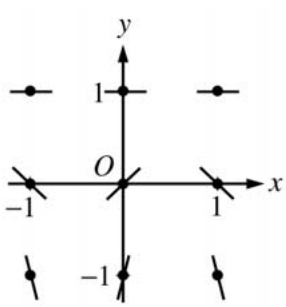
(a) On the axes provided, sketch a slope field for the given differential equation at the nine points indicated. **(Note: Use the axes provided in the exam booklet.)**



(b) There is a horizontal line with equation $y = c$ that satisfies this differential equation. Find the value of c.

(c) Find the particular solution $y = f(x)$ to the differential equation with the initial condition $f(1) = 0$.

(a)



$2 : \begin{cases} 1 : \text{zero slopes} \\ 1 : \text{all other slopes} \end{cases}$

(b) The line $y = 1$ satisfies the differential equation, so $c = 1$.

$1 : c = 1$

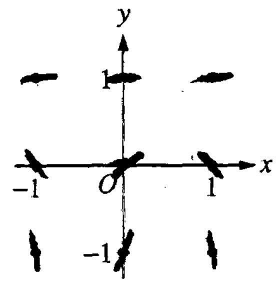
(c)
$$\frac{1}{(y-1)^2}\,dy = \cos(\pi x)\,dx$$
$$-(y-1)^{-1} = \frac{1}{\pi}\sin(\pi x) + C$$
$$\frac{1}{1-y} = \frac{1}{\pi}\sin(\pi x) + C$$
$$1 = \frac{1}{\pi}\sin(\pi) + C = C$$
$$\frac{1}{1-y} = \frac{1}{\pi}\sin(\pi x) + 1$$
$$\frac{\pi}{1-y} = \sin(\pi x) + \pi$$
$$y = 1 - \frac{\pi}{\sin(\pi x) + \pi} \quad \text{for } -\infty < x < \infty$$

$6 : \begin{cases} 1 : \text{separates variables} \\ 2 : \text{antiderivatives} \\ 1 : \text{constant of integration} \\ 1 : \text{uses initial condition} \\ 1 : \text{answer} \end{cases}$

Note: max $3/6$ [1-2-0-0-0] if no constant of integration
Note: $0/6$ if no separation of variables

Work for problem 5(a)

$$\frac{dy}{dx} = (y-1)^2 \cdot \cos \pi x.$$



Work for problem 5(b)

$y = C$

The horizontal line is $y=1$, because where $y=1$ the derivative of the the function is zero (it does not depend what x we take). And as equation of the linear function is $y=bx+c$ and the slope of it is zero, we may say that $\underline{y=1}$.

Continue problem 5 on page 13.

NO CALCULATOR ALLOWED

Work for problem 5(c)

$$\begin{cases} \dfrac{dy}{dx} = (y-1)^2 \cdot \cos(\pi x) \\ f(1) = 0. \end{cases}$$

$$\int \frac{dy}{(y-1)^2} = \int \cos(\pi x)\,dx$$

$$-\frac{1}{y-1} = \frac{\sin(\pi x)}{\pi} + C.$$

$$-\frac{1}{0-1} = \frac{\sin \pi}{\pi} + C.$$

$$1 = 0 + C.$$

$$\underline{C = 1}.$$

$$-\frac{1}{y-1} = \frac{\sin(\pi x)}{\pi} + 1.$$

$$\frac{1}{y-1} = -\frac{\sin(\pi x)}{\pi} - 1.$$

$$y-1 = -\frac{1}{\frac{\sin(\pi x)}{\pi} + 1}$$

$$\underline{y = -\frac{\pi}{\sin(\pi x) + \pi} + 1.}$$

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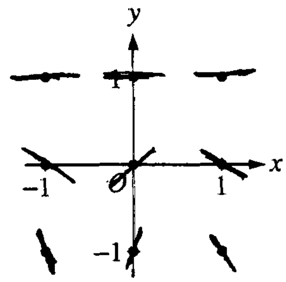
5B

NO CALCULATOR ALLOWED

Work for problem 5(a)

$(0-1)^2\cos(\pi(0)) = 1$

$(-1-1)^2\cos(\pi(0)) = 4$



$(0-1)^2\cos\pi = -1$

$(1-1)^2\cos\pi = 0$

$(0-1)^2\cos-\pi = -1$

$(-1-1)^2\cos-\pi = -4$

$(-1-1)^2\cos\pi = -4$

Work for problem 5(b)

$\frac{dy}{dx} = (y-1)^2\cos\pi x$ $\quad \int\frac{1}{(y-1)^2}\frac{dy}{dx} = \int\cos\pi x\,dx$

$-\frac{1}{y-1} = \frac{\sin\pi x}{\pi}$ $\quad y = -\frac{\pi}{\sin\pi x} + 1$

$y=1$ $\boxed{C=1}$

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Continue problem 5 on page 13.

Work for problem 5(c)

$u = y-1 \quad dx = \frac{du}{1}$

$\frac{dy}{dx} = (y-1)^2 \cos \pi x \qquad \int \frac{1}{(y-1)^2} \frac{dy}{dx} = \int \cos \pi x \, dx$

$-\frac{1}{y-1} = \frac{\sin \pi x}{\pi} \qquad y = -\frac{\pi}{\sin \pi x} + 1 + C$

$0 = \frac{-\pi}{\sin \pi} + 1 + C \qquad C = 0$

$\boxed{y = \frac{-\pi}{\sin \pi x}}$

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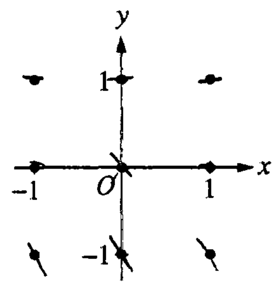
5C

NO CALCULATOR ALLOWED

Work for problem 5(a)



Work for problem 5(b)

$y' = 0$

$(y-1)^2 \cos(\pi x) = 0$

$=$

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Continue problem 5 on page 13.

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Work for problem 5(c)

$$\frac{1}{(y-1)^2}\,dy = \cos(\pi x)\,dx$$

$$\frac{-2}{y-1} = \sin(\pi x)(\pi) + C$$

$$\frac{-2}{(0)-1} = \sin(\pi(1))(\pi) + C$$

$$-2 - \pi\sin\pi = C$$

$$-2 = C$$

$$y - 1 = \frac{-2}{\pi\sin\pi - 2}$$

$$y = \frac{-2}{\pi\sin\pi - 2} + 1$$

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Question 5

Overview

This problem presented students with a separable differential equation. In part (a) students were asked to sketch its slope field at nine points. In part (b) students needed to recognize that if there is a horizontal line with equation $y = c$ that satisfies the differential equation, then $y = c$ must make $\frac{dy}{dx} = 0$ for all values of x. Part (c) required solving the separable differential equation to find the particular solution with $f(1) = 0$.

Sample: 5A
Score: 9

The student earned all 9 points.

Sample: 5B
Score: 6

The student earned 6 points: 2 points in part (a), 1 point in part (b), and 3 points in part (c). The work in parts (a) and (b) is correct. In part (c) the student correctly separates variables and finds the two antiderivatives, which earned 3 points. The student does not have a constant of integration so did not earn any of the last 3 points. The student does eventually add a constant but only after doing some algebraic simplification and thus at an inappropriate step in trying to find the particular solution.

Sample: 5C
Score: 4

The student earned 4 points: 1 point in part (a) and 3 points in part (c). In part (a) the zero slopes are correct, which earned 1 point. The slopes on the x-axis and at $(0, -1)$ are incorrect or missing. In part (b) the student does not complete the work and earned no points. In part (c) the student correctly separates variables and earned the first point. Both antiderivatives are incorrect. The student earned the third and fourth points with a correct introduction of a constant of integration and use of the initial condition. The sixth point was not earned because the student makes an error in solving for C.

Question 6

t (sec)	0	15	25	30	35	50	60
$v(t)$ (ft/sec)	–20	–30	–20	–14	–10	0	10
$a(t)$ $\left(\text{ft}/\text{sec}^2\right)$	1	5	2	1	2	4	2

A car travels on a straight track. During the time interval $0 \le t \le 60$ seconds, the car's velocity v, measured in feet per second, and acceleration a, measured in feet per second per second, are continuous functions. The table above shows selected values of these functions.

(a) Using appropriate units, explain the meaning of $\int_{30}^{60} |v(t)|\, dt$ in terms of the car's motion. Approximate $\int_{30}^{60} |v(t)|\, dt$ using a trapezoidal approximation with the three subintervals determined by the table.

(b) Using appropriate units, explain the meaning of $\int_{0}^{30} a(t)\, dt$ in terms of the car's motion. Find the exact value of $\int_{0}^{30} a(t)\, dt$.

(c) For $0 < t < 60$, must there be a time t when $v(t) = -5$? Justify your answer.

(d) For $0 < t < 60$, must there be a time t when $a(t) = 0$? Justify your answer.

(a) $\int_{30}^{60} |v(t)|\, dt$ is the distance in feet that the car travels from $t = 30$ sec to $t = 60$ sec.

Trapezoidal approximation for $\int_{30}^{60} |v(t)|\, dt$:

$$A = \frac{1}{2}(14+10)5 + \frac{1}{2}(10)(15) + \frac{1}{2}(10)(10) = 185 \text{ ft}$$

$2 : \begin{cases} 1 : \text{explanation} \\ 1 : \text{value} \end{cases}$

(b) $\int_{0}^{30} a(t)\, dt$ is the car's change in velocity in ft/sec from $t = 0$ sec to $t = 30$ sec.

$$\int_{0}^{30} a(t)\, dt = \int_{0}^{30} v'(t)\, dt = v(30) - v(0)$$
$$= -14 - (-20) = 6 \text{ ft/sec}$$

$2 : \begin{cases} 1 : \text{explanation} \\ 1 : \text{value} \end{cases}$

(c) Yes. Since $v(35) = -10 < -5 < 0 = v(50)$, the IVT guarantees a t in $(35, 50)$ so that $v(t) = -5$.

$2 : \begin{cases} 1 : v(35) < -5 < v(50) \\ 1 : \text{Yes; refers to IVT or hypotheses} \end{cases}$

(d) Yes. Since $v(0) = v(25)$, the MVT guarantees a t in $(0, 25)$ so that $a(t) = v'(t) = 0$.

$2 : \begin{cases} 1 : v(0) = v(25) \\ 1 : \text{Yes; refers to MVT or hypotheses} \end{cases}$

Units of ft in (a) and ft/sec in (b)

1 : units in (a) and (b)

NO CALCULATOR ALLOWED

6A1

t (sec)	0	15	25	30	35	50	60
$v(t)$ (ft/sec)	−20	−30	−20	−14	−10	0	10
$a(t)$ $\left(\text{ft}/\text{sec}^2\right)$	1	5	2	1	2	4	2

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Work for problem 6(a)

$\int_{30}^{60} |v(t)|\,dt$ is the ~~total~~ distance, in feet, travelled by the car from 30 to 60 seconds.

$$\int_{30}^{60} |v(t)|\,dt = \left|\tfrac{1}{2}(5)(-14-10)\right| + \left|\tfrac{1}{2}(15)(-10+0)\right| + \tfrac{1}{2}(10+0)(10)$$

$$= \frac{(24)(5)}{2} + \frac{(10)(15)}{2} + \frac{(10)(10)}{2}$$

$$= 60 + 75 + 50 = 185 \text{ feet.}$$

Work for problem 6(b)

$\int_{0}^{30} a(t)\,dt$ is the change in velocity from 0 seconds to 30 seconds (in feet /sec).

$$\int_{0}^{30} a(t)\,dt = v(30) - v(0) = -14 - (-20) = 6 \text{ ft/sec}$$

$\therefore$ there was an overall increase in the velocity of the car during the first 30 seconds equal to 6 ft/sec.

Continue problem 6 on page 15

6A2

Work for problem 6(c)

Yes, there must

Since $v(t)$ is continuous over $[35, 50]$ & differentiable,
& $v(t)$ is increasing & $a(t)$ is also increasing
during this time interval,
& since $v(t) = -5 \in (\overset{-10}{v(35)}, \overset{0}{v(50)}] \Rightarrow$
$\exists$ a time $t \in (35, 50) \mid v(t) = -5$ ft/sec

Work for problem 6(d)

Yes, there must.

for $0 < t < 60$, $\exists$ a $t \in (0, 60) \mid a(t) = 0$
by the mean value theorem
$v(0) = v(25) = -20$ ft/sec
$a(t) = v'(t) \Rightarrow$ since $v(t)$ is continuous & differentiable for $(0, 25) \Rightarrow$

$$\exists\ t \in (0, 25) \mid v'(t) = \frac{v(25) - v(0)}{25 - 0} = 0$$

$\therefore a(t) = 0$

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GO ON TO THE NEXT PAGE.

NO CALCULATOR ALLOWED

6B₁

t (sec)	0	15	25	30	35	50	60
$v(t)$ (ft/sec)	–20	–30	–20	–14	–10	0	10
$a(t)$ $\left(\text{ft}/\text{sec}^2\right)$	1	5	2	1	2	4	2

Work for problem 6(a)

$\int_{30}^{60} |v(t)|\,dt$ represents the total distance in feet travelled by the car between $t = 30$ and 60 seconds.

$$A_{tot.} = \left|\left(\frac{v(30)+v(35)}{2}\right)(35-30)\right| + \left|\left(\frac{v(50)+v(35)}{2}\right)(50-35)\right| + \left|\left(\frac{v(60)+v(50)}{2}\right)(60\right.$$

$$= \left|\left(\frac{-10-14}{2}\right)(5)\right| + \left|\left(\frac{-10}{2}(15)\right)\right| + \left|\frac{10}{2}(10)\right|$$

$$= 60 + 65 + 50 = \underline{175 \text{ feet.}}$$

Work for problem 6(b)

$\int_0^{30} a(t)\,dt$ represents the velocity of the car at $t = 30$ seconds in feet per second.

$$\underline{v(30) = -14 \text{ ft/s}}$$

Continue problem 6 on page 15.

6B2

Work for problem 6(c)

Yes; since $V(t)$ is a continuous function, it must pass through every value between -10 and 0 between $t = 35$ and $t = 50$

Work for problem 6(d)

Yes. Using Rolle's theorum, one finds that if the slope of a line between two points on a continuous function equals zero, than there must be some point, c, where the slope of that function is equal to zero.

$$\frac{-20-(-20)}{25-0} = 0$$

The line between the points at $t = 0$ and $t = 25$ is horizontal. It has zero slope so $a(t)$ (which equals $v'(t)$) must equal zero at some point between $t = 0$ and $t = 25$

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NO CALCULATOR ALLOWED

6C1

t (sec)	0	15	25	30	35	50	60
$v(t)$ (ft/sec)	–20	–30	–20	–14	–10	0	10
$a(t)$ (ft/sec^2)	1	5	2	1	2	4	2

Work for problem 6(a)

a) $\int_{30}^{60} |v(t)|\, dt$ is the total displacement of the car from 30 sec to 60 sec.

$$\int_{30}^{60} |v(t)|\,dt = \frac{(-14+-10)(5)}{2} + \frac{(-10+0)(15)}{2} + \frac{(10+0)(10)}{2} = -60 + -75 + 50$$
$$= -85$$

-85 ft

Work for problem 6(b)

b) $\int_0^{30} a(t)dt$ is the average velocity of the car from 0 sec to 30 sec.

$$\int_0^{30} a(t)\,dt = -14 - -20 = 6.$$

6 ft/sec

Continue problem 6 on page 15.

6C2

Work for problem 6(c)

c) yes, because of the intermediate value theorum. from $t=35$ to $t=50$ the velocity changes from -10 to 0, that means the velocity had to be at -5 some time between 35 and 50 because it is a continuous function.

Work for problem 6(d)

d) no, because the function is continuous and because of the intermediate value theorum, the acceleration was never at zero, because all the acceleration values are greater than zero.

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Question 6

Overview

This problem presented students with a table of the values of a car's velocity and acceleration at selected times. In part (a) students had to recognize the given definite integral as the total distance traveled by the car, in feet, from time $t = 30$ seconds to time $t = 60$ seconds and then approximate this distance using a trapezoidal approximation with three intervals of unequal lengths. In part (b) students had to recognize the given definite integral as the total change in velocity, in feet per second, from time $t = 0$ seconds to time $t = 30$ seconds and then calculate the exact value of this integral using the Fundamental Theorem of Calculus. Units of measure were important in both parts (a) and (b). In part (c) students were expected to use the Intermediate Value Theorem with $v(t)$ to justify that $v(t) = -5$ somewhere on the interval. Part (d) asked a similar question about the acceleration, but here students were expected to use the Mean Value Theorem applied to $v(t)$ to show the existence of a time t when $a(t) = v'(t) = 0$.

Sample: 6A
Score: 9

The student earned all 9 points.

Sample: 6B
Score: 6

The student earned 6 points: 1 point in part (a), 2 points in part (c), 2 points in part (d), and the units point. In part (a) the student earned the explanation point but makes an arithmetic mistake in the last line and so did not earn the value point. In part (b) the student understands that velocity is the antiderivative of acceleration but does not recognize the definite integral as the change in the velocity and did not earn the explanation point. The student does not find the correct value for the definite integral. In part (c) the student makes the correct conclusion and gives a correct reason, which earned both points. It is not necessary to name the Intermediate Value Theorem since the hypothesis (continuity) is mentioned. In part (d) the student applies Rolle's Theorem (the Mean Value Theorem is also acceptable). Although the student only mentions continuity in the general description of Rolle's Theorem, the second point was still earned. The correct units are used in parts (a) and (b), and the student therefore earned the units point.

Sample: 6C
Score: 4

The student earned 4 points: 1 point in part (b), 2 points in part (c), and the units point. In part (a) the "total displacement" is not the same as the total distance traveled, and so the student did not earn the first point. Because the student fails to use the absolute value in the first two terms of the trapezoidal approximation, the answer is incorrect. In part (b) the integral is not the "average velocity" so the student did not earn the first point. The computation for the second point is correct. In part (c) the student earned both points by correctly citing the Intermediate Value Theorem and drawing the correct conclusion. In part (d) the student answers "no." There is no way to justify an incorrect answer so the student earned no points in this part. The student gives the correct units in (a) and (b) and earned the units point.